Exercise 35

(a) For the limit $\lim_{x\to 1}(x^3 + x + 1) = 3$, use a graph to find a value of δ that corresponds to $\varepsilon = 0.4$.

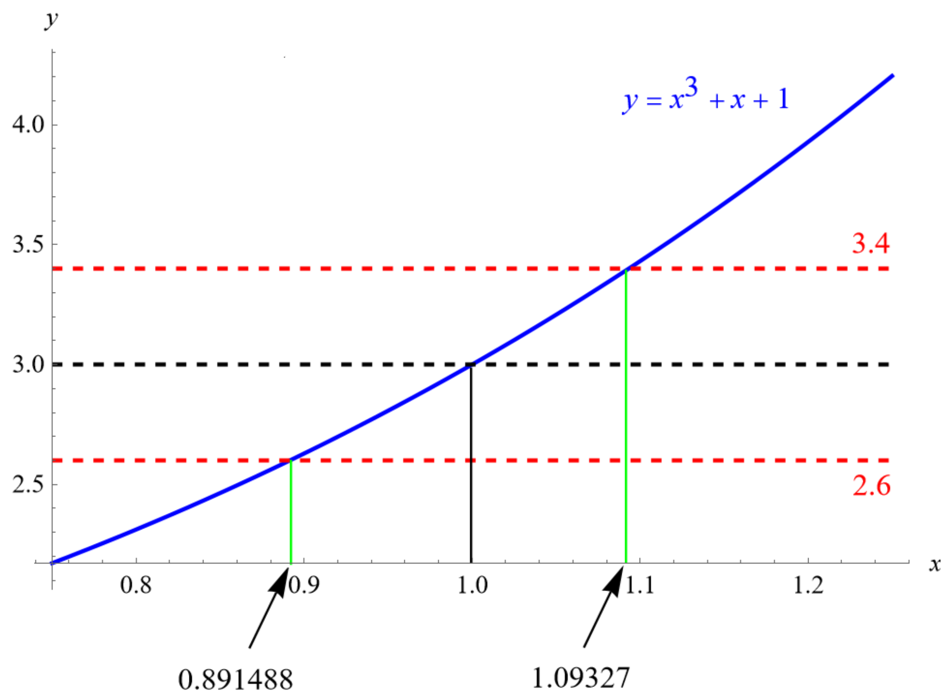
(b) By using a computer algebra system to solve the cubic equation $x^3 + x + 1 = 3 + \varepsilon$, find the largest possible value of δ that works for any given $\varepsilon > 0$.

(c) Put $\varepsilon = 0.4$ in your answer to part (b) and compare with your answer to part (a).

Solution

Part (a)

Draw a graph of $x^3 + x + 1$ versus x.

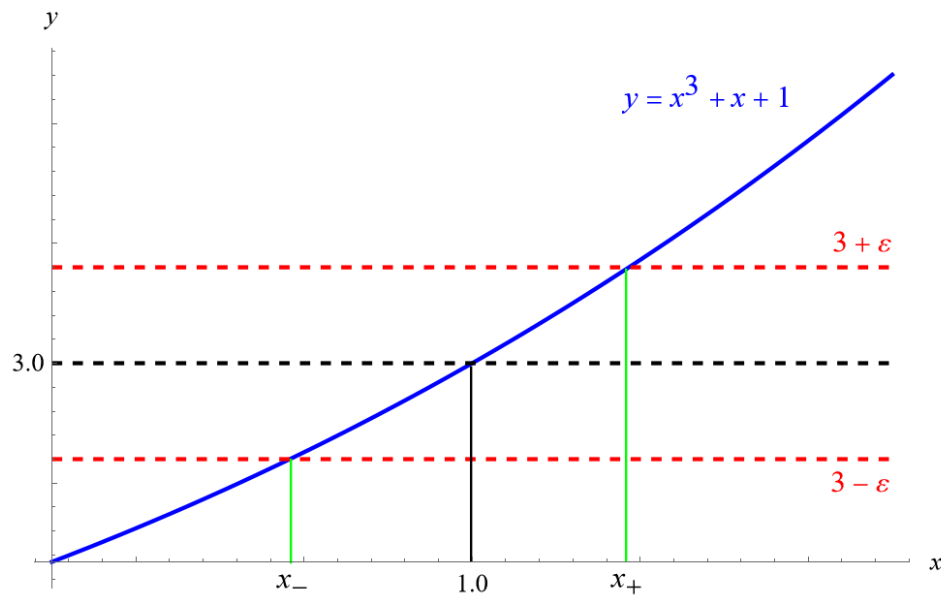


Because the graph of $x^3 + x + 1$ curves upward as x increases, the distance from 1 to 0.891488 is larger than the distance from 1 to 1.09327. The smaller distance is selected for δ so that the y-values remain between 2.6 and 3.4 as the x-values go between $1 - \delta$ and $1 + \delta$.

$$\delta \approx 0.09327$$

Part (b)

Draw a graph of $x^3 + x + 1$ versus x.



Use a computer algebra system to determine the x-values corresponding to $3 - \varepsilon$ and $3 + \varepsilon$.

$$x_-^3 + x_- + 1 = 3 - \varepsilon \qquad \rightarrow \qquad x_- = \frac{2^{1/3}\left[18 - 9\varepsilon + \sqrt{336 + 81\varepsilon(-4+\varepsilon)}\right]^{2/3} - 2 \times 3^{1/3}}{6^{2/3}\left[18 - 9\varepsilon + \sqrt{336 + 81\varepsilon(-4+\varepsilon)}\right]^{1/3}}$$

$$x_+^3 + x_+ + 1 = 3 + \varepsilon \qquad \rightarrow \qquad x_+ = \frac{2^{1/3}\left[18 + 9\varepsilon + \sqrt{336 + 81\varepsilon(4+\varepsilon)}\right]^{2/3} - 2 \times 3^{1/3}}{6^{2/3}\left[18 + 9\varepsilon + \sqrt{336 + 81\varepsilon(4+\varepsilon)}\right]^{1/3}}$$

As stated in part (a), the distance from 1 to x_+ is shorter than the distance from 1 to x_-, so we select

$$\delta = x_+ - 1 = \frac{2^{1/3}\left[18 + 9\varepsilon + \sqrt{336 + 81\varepsilon(4+\varepsilon)}\right]^{2/3} - 2 \times 3^{1/3}}{6^{2/3}\left[18 + 9\varepsilon + \sqrt{336 + 81\varepsilon(4+\varepsilon)}\right]^{1/3}} - 1$$

for any given $\varepsilon > 0$.

Part (c)

If $\varepsilon = 0.4$, then the result for δ in part (b) gives the result of part (a).

$$\delta = \frac{2^{1/3}\left[18 + 9(0.4) + \sqrt{336 + 81(0.4)(4+0.4)}\right]^{2/3} - 2 \times 3^{1/3}}{6^{2/3}\left[18 + 9(0.4) + \sqrt{336 + 81(0.4)(4+0.4)}\right]^{1/3}} - 1 \approx 0.09327$$